Notes on Midterm Exam

Math 31 – Abstract Algebra

The following is a list of topics that will potentially be covered on the midterm exam. It is basically a list of all the things that we have covered so far, and it corresponds roughly to Sections 0–5 and 7–10 (with only part of 8 covered so far) in Saracino's *Abstract Algebra: A First Course*. It would be difficult to cover all of these topics on the exam, but the questions will represent a large sample of the material, and you should be well–versed in all of them.

1. **Preliminaries:** Modular arithmetic and the Euclidean algorithm. One-to-one and onto mappings.
2. **Basic group theory:** Definition of groups and binary operations. Fundamental examples of groups, including $\mathbb{Z}$, $\mathbb{Z}_n$, $\mathrm{GL}_n(\mathbb{R})$, the symmetric group S_n, and the dihedral group D_n. Basic properties of groups. Cayley tables.
3. **Subgroups:** Definition of a subgroup. Basic examples. Criteria for checking that a subset is a subgroup. Cyclic subgroups.
4. **Cyclic groups:** Definition and basic properties of cyclic groups. Results on orders of elements and generators of cyclic groups.
5. **Lagrange's theorem:** Equivalence relations and cosets. Statement of Lagrange's theorem. Basic consequences of the theorem.

Exam Format

As you know, the midterm will be divided into two parts: an in–class exam, and a take–home exam. The in–class exam will largely test your understanding of the basic concepts that we have covered so far. Specifically, you should be prepared to state important definitions and theorems in a precise way. You may also be asked questions that require you to analyze a given statement, perhaps in a true–false format. There may also be computational questions, such as finding a greatest common divisor, or verifying that a given example is a subgroup. Finally, there will likely be a couple of simple proof questions. Since this is a timed exam, the proof questions will be fairly short and straightforward. To give you an idea of the sorts of things to expect, there will be a list of sample questions at the end of this document. The in-class exam will be closed-book, with no help given or received.

The take-home exam will be mostly proof-based. It will essentially be a homework assignment, albeit without the ability to collaborate. The questions will be of the same level of difficulty as the easy and medium questions on the regular homework. In fact, the medium-difficulty homework problems are meant to be indicative of the exam problems. You will be allowed to use your textbook, as well as the reserve text (*Contemporary Abstract Algebra* by Joseph Gallian). You may also use your lecture notes (including the notes that I have posted online) and your old homework assignments, but no other written materials are allowed (including the internet). You are also forbidden from discussing the exam with your classmates. I will not give any help other than answering questions regarding clarification of the exam problems.

Sample Questions for In-class Exam

The following is a list of practice proof questions for the in-class exam. These are meant to give you an idea of the expected level of difficulty of the exam questions. The problems that appear on the exam will not necessarily appear on this list, but these should illustrate the sorts of things that I think you should be able to prove in an exam setting.

1. Suppose that $*$ is an *associative* binary operation on a set S. Let

   $$H = \{a \in S : a * x = x * a \text{ for all } x \in S\}.$$

   Show that H is closed under $*$.

2. Let G be a group. Prove that the identity element $e \in G$ is unique.

3. Let G be a group. Prove that if $a, b \in G$, then $(ab)^{-1} = b^{-1}a^{-1}$.

4. Let a and b be elements of a group G, and suppose that

   $$(ab)^2 = a^2b^2.$$

   Show that $ab = ba$.

5. Let G be a group, and let H and K be subgroups of G. Show that $H \cap K$ is also a subgroup of G.

6. Prove that every cyclic group is abelian.

7. Show that a group that has only a finite number of subgroups must be a finite group.

8. Let G be a group of order pq, where p and q are prime numbers. Prove that every proper subgroup of G is cyclic.